TRINITY COLLEGE

**Semester One Examination, 2021**

**Question/Answer booklet**

# MATHEMATICS SPECIALIST UNIT 3

## Section Two: Calculator-assumed

# SOLUTIONS

WA student number: In figures

In words ______________________

Your name ______________________

## Time allowed for this section

Reading time before commencing work: ten minutes
Working time: one hundred minutes

Number of additional answer booklets used (if applicable):

## Materials required/recommended for this section

***To be provided by the supervisor***
This Question/Answer booklet
Formula sheet (retained from Section One)

***To be provided by the candidate***
Standard items: pens (blue/black preferred), pencils (including coloured), sharpener, correction fluid/tape, eraser, ruler, highlighters

Special items: drawing instruments, templates, notes on two unfolded sheets of A4 paper, and up to three calculators, which can include scientific, graphic and Computer Algebra System (CAS) calculators, are permitted in this ATAR course examination

## Important note to candidates

No other items may be taken into the examination room. It is **your** responsibility to ensure that you do not have any unauthorised material. If you have any unauthorised material with you, hand it to the supervisor **before** reading any further.

## Structure of this paper

| Section | Number of questions available | Number of questions to be answered | Working time (minutes) | Marks available | Percentage of examination |
|---|---|---|---|---|---|
| Section One: Calculator-free | 8 | 8 | 50 | 50 | 35 |
| Section Two: Calculator-assumed | 13 | 13 | 100 | 90 | 65 |
| | | | | **Total** | 100 |

## Instructions to candidates

1. The rules for the conduct of Trinity College examinations are detailed in the *Instructions to Candidates* distributed to students prior to the examinations. Sitting this examination implies that you agree to abide by these rules.

2. Write your answers in this Question/Answer booklet preferably using a blue/black pen. Do not use erasable or gel pens.

3. You must be careful to confine your answers to the specific question asked and to follow any instructions that are specific to a particular question.

4. Show all your working clearly. Your working should be in sufficient detail to allow your answers to be checked readily and for marks to be awarded for reasoning. Incorrect answers given without supporting reasoning cannot be allocated any marks. For any question or part question worth more than two marks, valid working or justification is required to receive full marks. If you repeat any question, ensure that you cancel the answer you do not wish to have marked.

5. It is recommended that you do not use pencil, except in diagrams.

6. Supplementary pages for planning/continuing your answers to questions are provided at the end of this Question/Answer booklet. If you use these pages to continue an answer, indicate at the original answer where the answer is continued, i.e. give the page number.

7. The Formula sheet is not to be handed in with your Question/Answer booklet.

**Section Two: Calculator-assumed** **65% (90 Marks)**

This section has **thirteen** questions. Answer **all** questions. Write your answers in the spaces provided.

Working time: 100 minutes.

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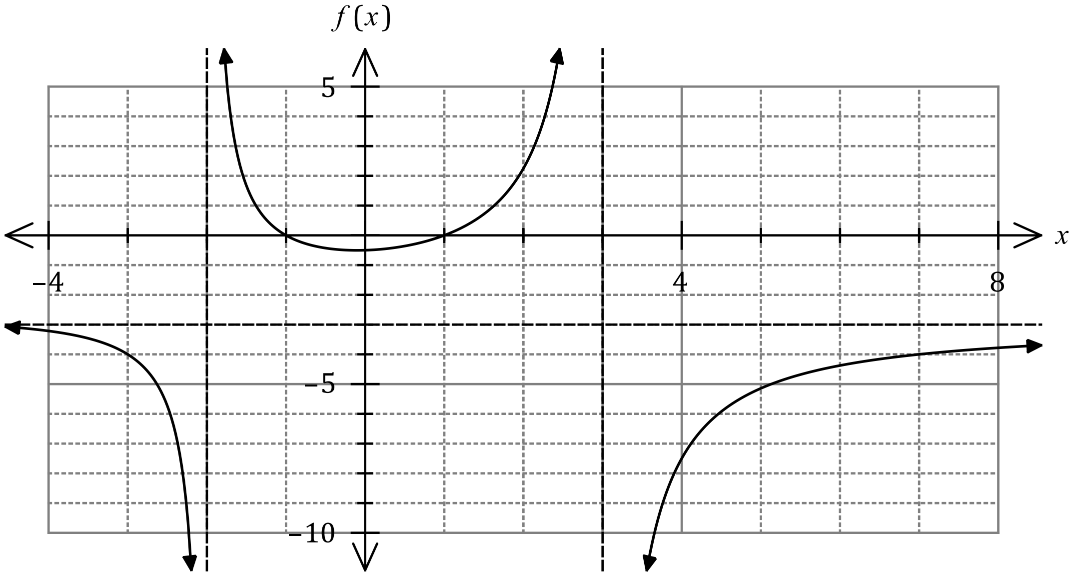
**Question 9** **(6 marks)**

The graph of $f(x) = \dfrac{a(b - x^2)}{(x - c)(x + d)}$ is shown below, where $a, b, c$ and $d$ are positive constants.

The dotted lines are the asymptotes of the function.




(a) Determine and write the value of each constant in the table below. (4 marks)

| Constant | $a$ | $b$ | $c$ | $d$ |
|---|---|---|---|---|
| Value | **3** | **1** | **3** | **2** |

| Solution |
|---|
| Pole at $x = -2 \Rightarrow d = 2$ |
| Pole at $x = 3 \Rightarrow c = 3$ |
| Roots at $x = \pm 1 \Rightarrow b = 1$ |
| Asymptote $y = -3 \Rightarrow a = 3$ |
| **Specific behaviours** |
| ✓ each constant |

(b) State the equations of all asymptotes of the graph of $y = \dfrac{1}{f(x)}$. (2 marks)

| Solution |
|---|
| Roots become vertical asymptotes: |
| $x = -1, \qquad x = 1$ |
| Horizontal asymptote: |
| $y = -\dfrac{1}{3}$ |
| **Specific behaviours** |
| ✓ both vertical asymptotes |
| ✓ horizontal asymptote |

**Question 10** **(7 marks)**

The arguments of the non-zero complex numbers $u$ and $v$ are $\theta$ and $\phi$ respectively, and the modulus of $u$ is twice the modulus of $v$.

Express the following in simplest form.

(a) $|u \div v|$. (1 mark)

| Solution |
|---|
| $|u| \div |v| = 2|v| \div |v| = 2$ |
| **Specific behaviours** |
| ✓ simplifies correctly |

(b) $\arg(iu) + \arg(\bar{u})$. (2 marks)

| Solution |
|---|
| $\arg(i) + \arg(u) - \arg(u) = \frac{\pi}{2}$ |
| **Specific behaviours** |
| ✓ indicates one correct simplification |
| ✓ simplifies correctly |

(c) $\frac{v\bar{v}}{|iu|}$. (2 marks)

| Solution |
|---|
| $\frac{|v|^2}{|i||u|} = \frac{|v|^2}{1 \times 2|v|} = \frac{1}{2}|v|$ |
| **Specific behaviours** |
| ✓ indicates two correct simplifications |
| ✓ simplifies correctly |

(d) $\arg\left(\frac{\overline{uv}}{3u^2}\right)$. (2 marks)

| Solution |
|---|
| $-\arg(u) - \arg(v) - 2\arg(u) = -3\theta - \phi$ |
| **Specific behaviours** |
| ✓ indicates two correct simplifications |
| ✓ simplifies correctly |

**Question 11** **(5 marks)**

The velocity vector of a particle at time $t$ seconds is given by $\mathbf{v}(t) = \begin{pmatrix} 2t-8 \\ 5 \\ 3e^{0.5t} \end{pmatrix}$ metres. The initial position vector of the particle is $18\mathbf{i} + 10\mathbf{j} + 2\mathbf{k}$.

(a) Determine the displacement vector $\mathbf{r}(t)$ for the particle after $t$ seconds. (3 marks)

**Solution**

$$\mathbf{r}(t) = \int \mathbf{v}(t)\,dt$$

$$= \begin{pmatrix} t^2-8t \\ 5t \\ 6e^{0.5t} \end{pmatrix} + \mathbf{c}$$

$$\mathbf{r}(0) = \begin{pmatrix} 0 \\ 0 \\ 6 \end{pmatrix} + \mathbf{c} = \begin{pmatrix} 18 \\ 10 \\ 2 \end{pmatrix} \Rightarrow \mathbf{c} = \begin{pmatrix} 18 \\ 10 \\ -4 \end{pmatrix}$$

$$\mathbf{r}(t) = \begin{pmatrix} t^2-8t+18 \\ 5t+10 \\ 6e^{0.5t}-4 \end{pmatrix}$$

**Specific behaviours**

✓ indicates need for integral of velocity vector
✓ correctly integrates components
✓ evaluates constant and writes displacement vector

(b) Determine the minimum distance of the particle from the $y$-$z$ plane. (2 marks)

**Solution**

Require $\mathbf{i}$-component to be minimum:

$$t^2 - 8t + 18 = (t-4)^2 + 2$$

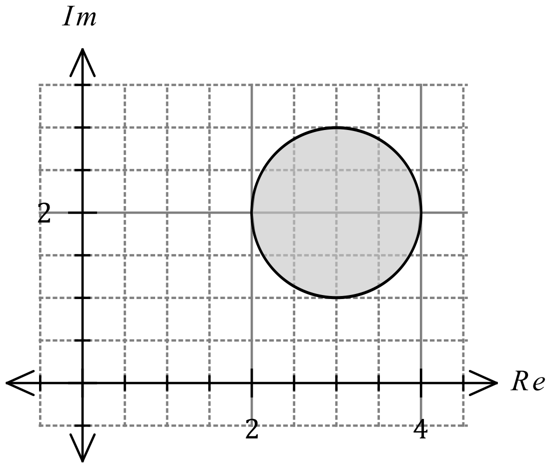
Hence minimum distance from $y$-$z$ plane is $2$ m.

**Specific behaviours**

✓ indicates $\mathbf{i}$-component to be minimum
✓ correct minimum distance

**Question 12** **(8 marks)**

(a) The locus of a complex number $z$ is the circular region shown below.



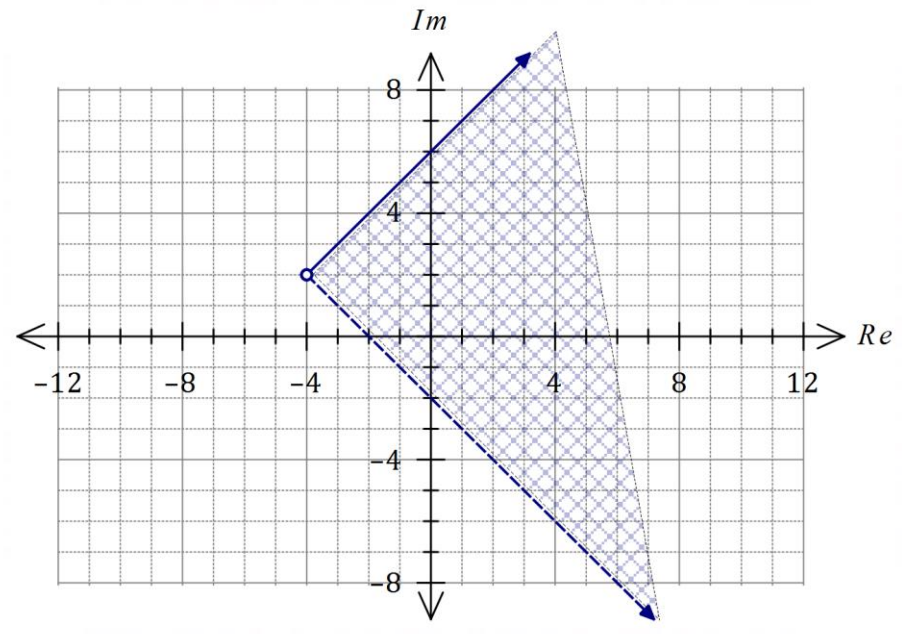

(i) Write equations or inequalities in terms of $z$ (without using $\text{Re}(z)$ or $\text{Im}(z)$) for the indicated locus. (3 marks)

| Solution |
|---|
| $\lvert z-(3+2i)\rvert \le 1$ |
| **Specific behaviours** |
| ✓ forms an inequality using modulus<br>✓ uses a difference between $z$ and circle centre<br>✓ uses correct radius on RHS |

(ii) Determine the minimum value for $\lvert z-3i\rvert$ as an exact value. (2 marks)

| Solution |
|---|
| Let line from $A$ (at $3i$) to circle centre $O$ intersect the circle at $P$.<br>$\lvert OA\rvert=\sqrt{3^2+1^2}=\sqrt{10}$<br>$\lvert AP\rvert=\lvert OA\rvert-r$<br>$=\sqrt{10}-1$ |
| **Specific behaviours** |
| ✓ indicates how minimum occurs<br>✓ calculates correct exact minimum |

(b) On the complex plane below sketch the locus of the complex number $z$ determined by $-\frac{\pi}{4} < \arg(z + 4 - 2i) \leq \frac{\pi}{4}$. (3 marks)




| Solution |
|---|
| See graph (allow filled point for start of ray) |
| **Specific behaviours** |
| ✓ uses correct translation from origin for start of rays<br>✓ lower ray dashed and angled correctly<br>✓ upper ray solid, angled correctly; correct shading |

**Question 13** **(7 marks)**

Functions $f$ and $g$ are defined as $f(x) = \dfrac{1}{\sqrt{x+1}}$ and $g(x) = e^{x^2-1}$.

(a) State the domain of $f(x)$ and explain why $f$ has an inverse. (2 marks)

| Solution |
|---|
| $D_f = \{x: x \in \mathbb{R}, x > -1\}$<br><br>$f$ has an inverse as it is a one-to-one function. |
| **Specific behaviours** |
| ✓ correct domain<br>✓ states $f$ is one-to-one |

(b) Determine the defining rule for $f^{-1}(x)$ and state its range. (2 marks)

| Solution |
|---|
| $x = \dfrac{1}{\sqrt{y+1}} \to y + 1 = \dfrac{1}{x^2} \to f^{-1}(x) = \dfrac{1}{x^2} - 1$<br><br>$R_{f^{-1}} = D_f = \{y: y \in \mathbb{R}, y > -1\}$ |
| **Specific behaviours** |
| ✓ correct defining rule<br>✓ correct range |

(c) Determine the defining rule for $g\big(f(x)\big)$ and state its domain and range. (3 marks)

| Solution |
|---|
| $g\big(f(x)\big) = e^{\frac{1}{x+1}-1}$<br><br>Domain: $\{x: x \in \mathbb{R}, x > -1\}$<br><br>Range: As $x \to -1^+, g\big(f(x)\big) \to \infty$ and as $x \to \infty, g\big(f(x)\big) \to e^{-1}$<br><br>$\left\{y: y \in \mathbb{R}, y > \dfrac{1}{e}\right\}$ |
| **Specific behaviours** |
| ✓ simplified composite function<br>✓ correct domain<br>✓ correct range |

**Question 14** **(7 marks)**

The position vectors of the points $A$ and $B$ are $\mathbf{r}_A = \begin{pmatrix} 5 \\ 2 \\ 3 \end{pmatrix}$ and $\mathbf{r}_B = \begin{pmatrix} 1 \\ -4 \\ 5 \end{pmatrix}$.

(a) If line segment $AB$ is the diameter of sphere $S$, determine the vector equation of $S$. (3 marks)

| Solution |
|---|
| Centre of sphere: $\overrightarrow{OC} = \frac{1}{2}(\mathbf{r}_A + \mathbf{r}_B) = \begin{pmatrix} 3 \\ -1 \\ 4 \end{pmatrix}$ <br> Radius of sphere: $r = \left\lvert \begin{pmatrix} 5 \\ 2 \\ 3 \end{pmatrix} - \begin{pmatrix} 3 \\ -1 \\ 4 \end{pmatrix} \right\rvert = \sqrt{14}$ <br> Equation: $\left\lvert \mathbf{r} - \begin{pmatrix} 3 \\ -1 \\ 4 \end{pmatrix} \right\rvert = \sqrt{14}$ |
| **Specific behaviours** |
| ✓ calculates position vector of centre <br> ✓ calculates radius <br> ✓ correct vector equation |

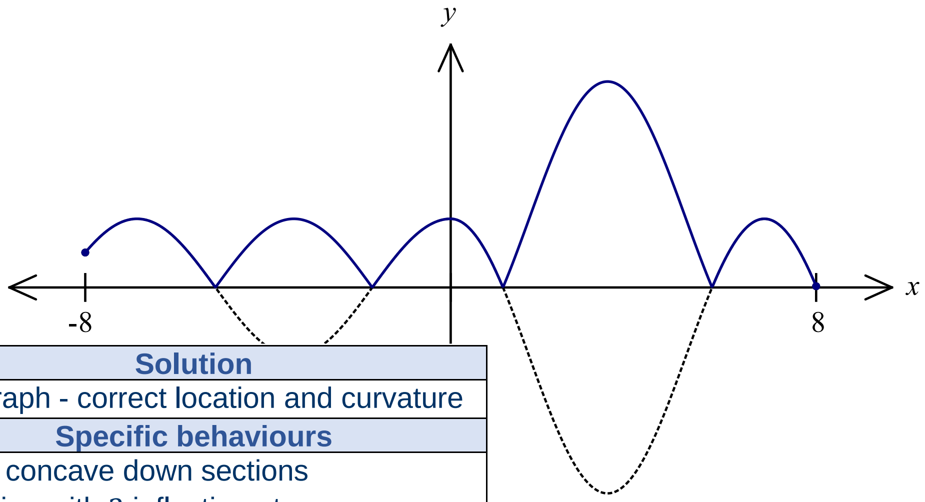
Straight line $L$ intersects the surface of sphere $S$ at point $A$ and has equation $\mathbf{r} = \mathbf{r}_A + \lambda \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix}$.

(b) Determine the position vector of $C$, the other point of intersection of $L$ with $S$. (4 marks)

| Solution |
|---|
| $\mathbf{r} = \begin{pmatrix} 5 \\ 2 \\ 3 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix}$ <br> Substitute line into sphere: $\left\lvert \begin{pmatrix} 5+\lambda \\ 2+2\lambda \\ 3-\lambda \end{pmatrix} - \begin{pmatrix} 3 \\ -1 \\ 4 \end{pmatrix} \right\rvert = \sqrt{14}$ <br> Simplify and solve: $(2+\lambda)^2 + (3+2\lambda)^2 + (-1-\lambda)^2 = 14$ <br> $6\lambda^2 + 18\lambda = 0$ <br> $\lambda = 0, \quad \lambda = -3$ <br> Other point, $C$: $\mathbf{r}_C = \begin{pmatrix} 5 \\ 2 \\ 3 \end{pmatrix} - 3\begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix} = \begin{pmatrix} 2 \\ -4 \\ 6 \end{pmatrix}$ |
| **Specific behaviours** |
| ✓ substitutes line into sphere <br> ✓ simplifies equation to remove magnitude <br> ✓ solves for $\lambda$ <br> ✓ correct position vector |

**Question 15** **(7 marks)**

(a) The graph of $y = f(x)$ is shown with a dotted line on the axes below.



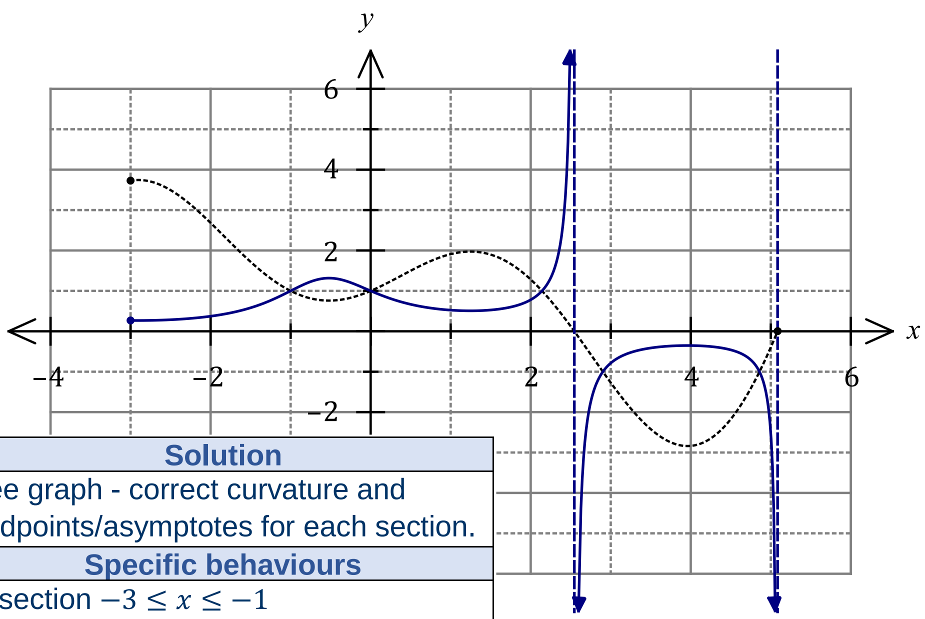

| Solution |
|---|
| See graph - correct location and curvature |
| **Specific behaviours** |
| ✓ four concave down sections |
| ✓ section with 2 inflection pts |

(i) On the same axes, sketch the graph of $y = |f(x)|$. (2 marks)

(ii) State the number of roots that the graph $y = f(|x|)$ will have. (1 mark)

| Solution |
|---|
| It will have $2 \times 3 = 6$ roots. |
| **Specific behaviours** |
| ✓ correct number of roots |

(b) The graph of $y = g(x)$ is shown with a dotted line on the axes below. Sketch the graph of $y = \dfrac{1}{g(x)}$ on the same axes. (4 marks)




| Solution |
|---|
| See graph - correct curvature and endpoints/asymptotes for each section. |
| **Specific behaviours** |
| ✓ section $-3 \le x \le -1$ |
| ✓ section $-1 < x \le 0$ |
| ✓ section $x > 0$ to asymptote |
| ✓ remaining section, below axis |

**Question 16** **(8 marks)**

(a) One solution to the equation $z^3 = u$ is $z = 2\,\text{cis}(-40°)$.

(i) Determine the other two solutions, giving solutions in the form $r\,\text{cis}\,\theta$, where $r \geq 0$ and $-180° < \theta \leq 180°$. (2 marks)

| Solution |
|---|
| $360° \div 3 = 120°$ |
| $z_2 = 2\,\text{cis}(-160°)\,, \qquad z_3 = 2\,\text{cis}(80°)$ |
| **Specific behaviours** |
| ✓ indicates angle separation of solutions |
| ✓ both solutions in required form |

(ii) Determine $u$, giving your answer in the form $a + bi$. (2 marks)

| Solution |
|---|
| $u = (2\,\text{cis}(-40°))^3$ |
| $= 8\,\text{cis}(-120°)$ |
| $= -4 - 4\sqrt{3}i$ |
| **Specific behaviours** |
| ✓ $u$ in polar form |
| ✓ $u$ in required form |

(b) Solve the equation $z^5 = 16\sqrt{2} - 16\sqrt{2}i$, giving exact solutions in the form $r\,\text{cis}\,\theta$, where $r \geq 0$ and $-\pi < \theta \leq \pi$. (4 marks)

| Solution |
|---|
| $z^5 = 16\sqrt{2} - 16\sqrt{2}i$ |
| $= 32\,\text{cis}\left(-\frac{\pi}{4}\right)$ |
| $z_k = 2\,\text{cis}\left(-\frac{\pi}{20} + \frac{2k\pi}{5}\right), k \in \mathbb{Z}$ |
| $z_0 = 2\,\text{cis}\left(-\frac{\pi}{20}\right)$ |
| $z_1 = 2\,\text{cis}\left(\frac{7\pi}{20}\right)$ |
| $z_2 = 2\,\text{cis}\left(\frac{15\pi}{20}\right)$ |
| $z_3 = 2\,\text{cis}\left(-\frac{17\pi}{20}\right)$ |
| $z_4 = 2\,\text{cis}\left(-\frac{9\pi}{20}\right)$ |
| **Specific behaviours** |
| ✓ writes equation in polar form |
| ✓ indicates one correct root |
| ✓ indicates angle separation of roots |
| ✓ all roots in required form |

**Question 17** **(8 marks)**

Four points in space have coordinates $A(5,4,-5), B(6,3,-2), C(0,-1,5)$ and $D(-3,0,1)$.

(a) Show that the lines $AC$ and $BD$ intersect and determine the coordinates of their point of intersection. (5 marks)

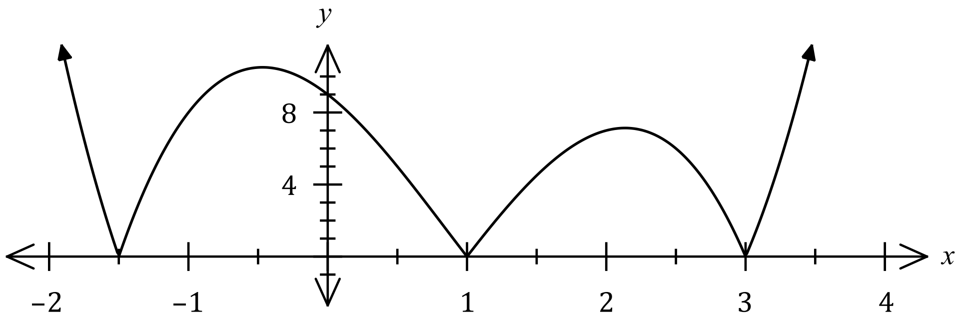
| Solution |
|---|
| $\overrightarrow{AC} = \begin{pmatrix} 0-5 \\ -1-4 \\ 5+5 \end{pmatrix} = \begin{pmatrix} -5 \\ -5 \\ 10 \end{pmatrix}, \quad \mathbf{r}_A = \begin{pmatrix} 5 \\ 4 \\ -5 \end{pmatrix} + t\begin{pmatrix} -1 \\ -1 \\ 2 \end{pmatrix}$ |
| $\overrightarrow{BD} = \begin{pmatrix} -3-6 \\ 0-3 \\ 1+2 \end{pmatrix} = \begin{pmatrix} -9 \\ -3 \\ 3 \end{pmatrix}, \quad \mathbf{r}_B = \begin{pmatrix} 6 \\ 3 \\ -2 \end{pmatrix} + s\begin{pmatrix} -3 \\ -1 \\ 1 \end{pmatrix}$ |
| Using $\mathbf{i}, \mathbf{j}$ coefficients: |
| $5 - t = 6 - 3s$ <br> $4 - t = 3 - s$ <br> $s = 1, t = 2$ |
| Check with $\mathbf{k}$ coefficients: |
| $-5 + 4 = -1 \text{ and } -2 + 1 = -1$ |
| Hence solution consistent with all three coefficients and so lines intersect at a point: |
| $\left.\begin{pmatrix} 6 \\ 3 \\ -2 \end{pmatrix} + s\begin{pmatrix} -3 \\ -1 \\ 1 \end{pmatrix}\right\rvert_{s=1} = \begin{pmatrix} 3 \\ 2 \\ -1 \end{pmatrix}$ |
| Point of intersection is at $(3, 2, -1)$. |
| **Specific behaviours** |
| ✓ equation for one line <br> ✓ equation for second line <br> ✓ writes set of simultaneous equations and solves for $s$ and $t$ <br> ✓ checks for consistency and infers intersection <br> ✓ calculates point of intersection |

(b) Determine the Cartesian equation of the plane containing the four points. (3 marks)

| Solution |
|---|
| Directions of lines in plane: $\begin{pmatrix} 1 \\ 1 \\ -2 \end{pmatrix}$ and $\begin{pmatrix} 3 \\ 1 \\ -1 \end{pmatrix}$ |
| Normal to plane: |
| $\begin{pmatrix} 1 \\ 1 \\ -2 \end{pmatrix} \times \begin{pmatrix} 3 \\ 1 \\ -1 \end{pmatrix} = \begin{pmatrix} 1 \\ -5 \\ -2 \end{pmatrix}$ |
| Hence equation is |
| $x - 5y - 2z = \begin{pmatrix} 1 \\ -5 \\ -2 \end{pmatrix} \cdot \begin{pmatrix} 3 \\ 2 \\ -1 \end{pmatrix} = -5$ |
| **Specific behaviours** |
| ✓ indicates two vectors in plane <br> ✓ obtains normal to plane <br> ✓ states equation of plane |

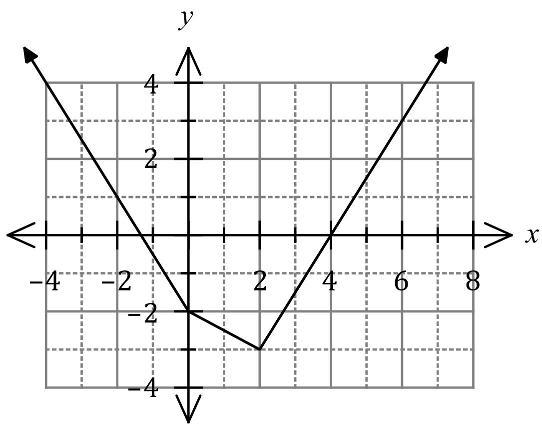
**Question 18** **(6 marks)**

(a) The graph of $y = |f(x)|$ is shown below, where $f(x) = ax^3 + bx^2 + cx - 9$. Determine the value of each of the coefficients $a, b$ and $c$. (3 marks)




| Solution |
|---|
| From roots, $f(x) = a(x + 1.5)(x - 1)(x - 3)$.<br><br>From given function, $f(0) = -9 \Rightarrow 4.5a = -9 \Rightarrow a = -2$.<br><br>$f(x) = -(2x + 3)(x^2 - 4x + 3) = -(2x^3 - 5x^2 - 6x + 9)$<br><br>Hence $a = -2, b = 5$ and $c = 6$. |
| **Specific behaviours** |
| ✓ uses roots to obtain factored form of $f(x)$ with constant $a$<br>✓ correct value of $a$<br>✓ correct values of $b$ and $c$ |

(b) The graph of $y = |px| + |x + q| + r$ is shown below, where $p, q$ and $r$ are constants. Determine the possible value(s) of each constant. (3 marks)

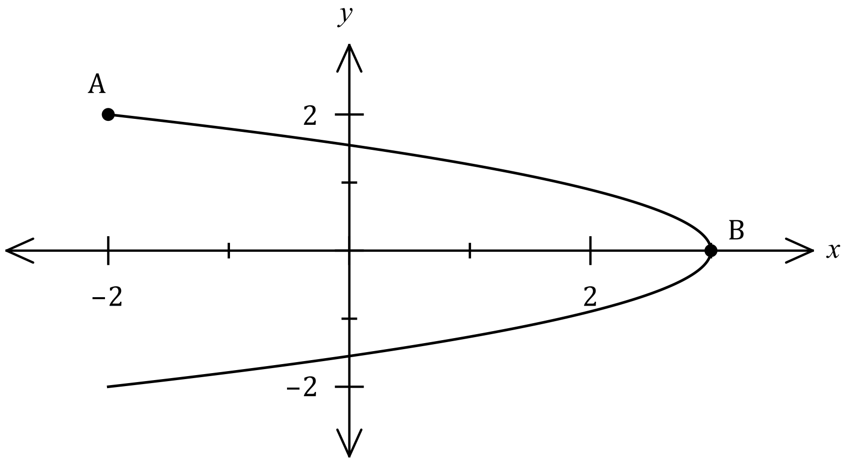



| Solution |
|---|
| 'Kink' at $x = 2 \Rightarrow q = -2$.<br><br>When $x > 2$, slope is $\frac{3}{2}$.<br>Hence $|p| + 1 = \frac{3}{2} \Rightarrow p = \pm 0.5$.<br><br>Using $(0, -2) \rightarrow 0 + 2 + r = -2 \Rightarrow r = -4$ |
| **Specific behaviours** |
| ✓ value of $q$<br>✓ value of $r$<br>✓ both values of $p$ |

**Question 19** **(8 marks)**

The position vector of a particle at time $t$ seconds is given by $\mathbf{r}(t) = \begin{pmatrix} 5\sin^2(t) - 2 \\ 2\cos t \end{pmatrix}$ cm.

The path of the particle is shown below, together with the points $A(-2, 2)$ and $B(3, 0)$ that lie on its path.




(a) Express the path of the particle as a Cartesian equation. (3 marks)

| Solution |
|---|
| Note domain restriction: $-2 \le x \le 3$ <br> $\frac{x+2}{5} = \sin^2 t\,, \quad \left(\frac{y}{2}\right)^2 = \cos^2 t$ <br> Hence <br> $\frac{x+2}{5} + \frac{y^2}{4} = 1, \quad -2 \le x \le 3$ |
| **Specific behaviours** |
| ✓ indicates use of Pythagorean identity to eliminate $t$ <br> ✓ obtains Cartesian equation <br> ✓ includes domain or range restriction |

(b) Determine the velocity of the particle when $t = \frac{\pi}{4}$. (2 marks)

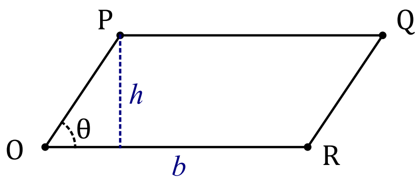
| Solution |
| --- |
| $\mathbf{v}(t) = \begin{pmatrix} 10\cos t \sin t \\ -2\sin t \end{pmatrix}$ <br> $\mathbf{v}\left(\frac{\pi}{4}\right) = \begin{pmatrix} 5 \\ -\sqrt{2} \end{pmatrix} \approx \begin{pmatrix} 5 \\ -1.41 \end{pmatrix}$ |
| **Specific behaviours** |
| ✓ obtains velocity vector <br> ✓ indicates velocity at given time |

(c) Determine the distance travelled by the particle as it moves from $A$ to $B$. (3 marks)

| Solution |
| --- |
| Particle is at $A$ when $t = 0$ and at $B$ when $t = \frac{\pi}{2}$. <br> $d = \int_0^{\frac{\pi}{2}} \lvert\mathbf{v}(t)\rvert \, dt$ <br> $= \int_0^{\frac{\pi}{2}} \sqrt{(10\cos t \sin t)^2 + (-2\sin t)^2} \, dt$ <br> $= 5.56$ cm |
| **Specific behaviours** |
| ✓ indicates correct bounds for integral <br> ✓ indicates expression for speed <br> ✓ correct distance |

**Question 20** **(6 marks)**

In the parallelogram shown, $\overrightarrow{OP} = \mathbf{p}$, $\overrightarrow{OR} = \mathbf{r}$ and the angle between the directions of $\mathbf{p}$ and $\mathbf{r}$ is $\theta$.




It can be shown that $|\mathbf{p} \times \mathbf{r}| = |\mathbf{p}||\mathbf{r}|\sin\theta$.

(a) Explain why evaluating $|\mathbf{p} \times \mathbf{r}|$ will result in the area of the parallelogram. (2 marks)

| Solution |
|---|
| Area is length of base ($|\mathbf{r}|$) multiplied by perpendicular height ($|\mathbf{p}|\sin\theta$): $$A = b \times h = |\mathbf{r}| \times |\mathbf{p}|\sin\theta = |\mathbf{r} \times \mathbf{p}|$$ |
| **Specific behaviours** |
| ✓ relates one vector to base<br>✓ relates other vector and angle to perpendicular height |

The area of $OPQR$ is $3\sqrt{5}$ cm$^2$ when the position vectors of $O, P$ and $Q$ are $\begin{pmatrix}0\\0\\0\end{pmatrix}, \begin{pmatrix}a\\-2\\1\end{pmatrix}$ and $\begin{pmatrix}3\\-2\\-1\end{pmatrix}$ respectively, with units in centimetres.

(b) Determine the value(s) of the constant $a$. (4 marks)

| Solution |
|---|
| Determine $\mathbf{r}$: $$\overrightarrow{OR} = \overrightarrow{OQ} - \overrightarrow{OP}$$ $$= \begin{pmatrix}3\\-2\\-1\end{pmatrix} - \begin{pmatrix}a\\-2\\-1\end{pmatrix} = \begin{pmatrix}3-a\\0\\-2\end{pmatrix}$$ Evaluate cross product (CAS): $$\mathbf{p} \times \mathbf{r} = \begin{pmatrix}a\\-2\\1\end{pmatrix} \times \begin{pmatrix}3-a\\0\\-2\end{pmatrix} = \begin{pmatrix}4\\3+a\\6-2a\end{pmatrix}$$ Magnitude of cross product (CAS): $$|\mathbf{p} \times \mathbf{r}| = \sqrt{5a^2 - 18a + 61}$$ Equate to area and solve (CAS): $$5a^2 - 18a + 61 = 45$$ $$a = 2, \qquad a = \frac{8}{5}$$ |
| **Specific behaviours** |
| ✓ vector for $\mathbf{r}$<br>✓ indicates cross product<br>✓ indicates magnitude of cross product<br>✓ equates to area and solves for correct values |

**Question 21** **(7 marks)**

Let the complex number $z = \sqrt{3} + i$ and the function $f$ be defind as $f(n) = (z)^n - (\bar{z})^n, n \in \mathbb{Z}$.

(a) Determine the modulus and argument of $f(-1)$. (2 marks)

| Solution |
|---|
| $f(-1) = -\frac{i}{2} = \frac{1}{2}\text{cis}\left(-\frac{\pi}{2}\right)$<br><br>Hence $\lvert f(-1)\rvert = \frac{1}{2}$ and $\arg\big(f(-1)\big) = -\frac{\pi}{2}$. |
| **Specific behaviours** |
| ✓ indicates correct $f(-1)$<br>✓ clearly states both modulus and argument |

(b) Use De Moivre's theorem to determine all values of $n$ for which $f(n) = 0$. (5 marks)

| Solution |
|---|
| $(\sqrt{3}+i)^n - (\sqrt{3}-i)^n = 0$<br>$\left(2\,\text{cis}\left(\frac{\pi}{6}\right)\right)^n - \left(2\,\text{cis}\left(-\frac{\pi}{6}\right)\right)^n = 0$<br>$2^n\left(\text{cis}\left(\frac{n\pi}{6}\right) - \text{cis}\left(-\frac{n\pi}{6}\right)\right) = 0$<br>$\cos\left(\frac{n\pi}{6}\right) + i\sin\left(\frac{n\pi}{6}\right) - \cos\left(-\frac{n\pi}{6}\right) - i\sin\left(-\frac{n\pi}{6}\right) = 0$<br>$\cos\left(\frac{n\pi}{6}\right) + i\sin\left(\frac{n\pi}{6}\right) - \cos\left(\frac{n\pi}{6}\right) + i\sin\left(\frac{n\pi}{6}\right) = 0$<br>$2i\sin\left(\frac{n\pi}{6}\right) = 0$<br>$\sin\left(\frac{n\pi}{6}\right) = 0$<br>$n = 6k, \quad k \in \mathbb{Z}$ |
| **Specific behaviours** |
| ✓ equation in polar form<br>✓ applies De Moivre's theorem<br>✓ fully expands polar form, adjusting terms for positive arguments<br>✓ simplifies to single term<br>✓ states all possible values |

**End of questions**

Supplementary page

Question number: _________

Supplementary page

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